International Journal of Structural Stability and Dynamics
Vol. 19, No. 3 (2019) 1950036 (17 pages)
© World Scientific Publishing Company
DOI: 10.1142/S0219455419500366

Buckling Analysis of Nanobeams Based on Nonlocal Timoshenko Beam Model by the Method of Initial Values

Erol Demirkan

Department of Civil Engineering, Technical University of Istanbul
Maslak-Istanbul 34469, Turkey
edemirkan@itu.edu.tr

Reha Artan*

Department of Mechanical Engineering
Friedrich-Alexander-Universität, Erlangen-Nürnberg 91054, Germany
artanr@itu.edu.tr; rartanitu@gmail.com

Received 4 June 2018
Accepted 28 October 2018
Published 10 December 2018

Investigated herein is the buckling of nanobeams based on a nonlocal Timoshenko beam model by the method of initial values within the framework of nonlocal elasticity. Since the nonlocal Timoshenko beam theory is of higher order than the nonlocal Euler–Bernoulli beam theory, it is known to be superior in predicting the small-scale effect. The buckling determinants and critical loads for bars with various kinds of supports are presented. The Carry-Over matrix (Transverse Matrix) is presented and the priorities of the method of initial values are depicted. To the best of the researchers' knowledge, this is the first work that investigates the buckling of nonlocal Timoshenko beam with the method of initial values.

Keywords: Buckling of Timoshenko beam; nonlocal elasticity; method of initial values.

1. Introduction

The Timoshenko beam theory allows the rotation between the cross-section and the bending line. This deformation is related to shear rotation, which is not considered in a Bernoulli beam. As a result, shear strains and stresses are removed from the Bernoulli beam theory. Therefore, Bernoulli beam is powerful. The Timoshenko beam theory was presented in the early part of the 20th century. Nowadays, the buckling of Timoshenko beams is investigated by experimental and numerical methods.[1] Wu *et al.*[2] investigate the free vibration and elastic buckling of sandwich

---

[*] Corresponding author.

beams with a stiff core and functionally graded carbon nanotube reinforced composite (FG-CNTRC) face sheets within the framework of Timoshenko beam theory.

Today, interest in nanotechnology is increasing strongly, and many nanostructures are produced. A nanostructure is a small object of intermediate size between molecular and microscopic (micrometer-sized) structures. Structural elements such as beams, plates, and membranes in micro- or nanolength scale are widely used as components in micro- or nanoelectromechanical system (MEMS or NEMS) devices.[3–6] To better understand the mechanical behavior of these structures; small-scale effects should be considered. The classical continuum theories are not capable of capturing the small-scale effects when dealing with nanoscale structures. In 1972, Eringen[7,8] presented the nonlocal continuum mechanics to take into account the small-scale effect. According to this theory, the stress at a reference point in a body depends not only on the strains on that point, but also on strains on all other points of the body. Peddieson *et al.*[9] first applied the nonlocal continuum theory to the nanotechnology in which the static deformations of beam structures were obtained by using a simplified nonlocal beam model based on the nonlocal elasticity theory of Eringen.[10] Later, many researchers examined the bending, buckling, vibration and wave propagation problems using the nonlocal elasticity theory.[11–19] Wang and Hu[20] studied flexural wave propagation in single-walled carbon nanotubes (SWCNTs) and presented that only the nonlocal elastic Timoshenko beam is able to predict the decrease in phase velocity when the wave number is so large that the microstructure of carbon nanotubes has a significant influence on the flexural wave dispersion. Wang *et al.*[21] used the principle of virtual work to obtain the governing equations and the boundary conditions for the buckling of axially loaded rods/tubes based on the nonlocal Timoshenko beam theory. In this paper, the method of initial values (transfer matrix method) is used to obtain the critical loads of rods with various end conditions in the frame of nonlocal elasticity. This method gives the values of the displacements and stress resultants throughout the rod once the initial displacements and initial stress resultants are known. A priority of the method is that the high degree of statical indeterminacy adds no additional hardship to the solution of the problem.[4,22]

2. The Physical Meaning of the Nonlocality Parameter and a Short Review of Nonlocal Elastic Models in Modeling of Carbon Nanotubes and Graphenes

Nonlocality parameter can have different physical meanings: a distance between particles in discrete structures, the dimension of the grain or a cell, a characteristic radius of correlation or action — at — a distance forces, etc. However, we shall always assume that the scale parameters are small in comparison with the dimensions of the body.[23] According to the aforementioned studies on the basis of Eringen's nonlocal elasticity theory, a consensus has been reached that the refined nonlocal models are superior to their local counterparts. However, since the theory also belongs to the category of continuum mechanics, verification of nonlocal continuum models is essential to identify the magnitude of the small-scale parameter γ which is the key revealing scale effect in models.[24] The nonlocality parameter revealing the small-scale effect on the responses of structures of nanosize. In recent years, a large number of studies have been conducted to determine the nonlocality parameter.[25,26] Generally, a conservative estimate of the nonlocality parameter is $\gamma < 2\,\text{nm}$ for a SWCNT for wave propagation.[27] Wang[18] indicated that nonlocal elasticity theory should be used for an accurate prediction of wave propagation in carbon nanotubes (CNTs), and estimated a range of the small-scale parameter. Khademolhosseini[28] examined the vibration of CNTs with nonlocal elasticity formulation and molecular dynamics simulation, showing that both models are in great agreement for nonlocality parameter (γ) is 0.185 nm. Liang and Han[29] show that nonlocal scale parameter is independent of the length of CNTs, and dependent on the radius of CNTs. In Han and Weng's paper,[26] the length-dependent stiffness is revealed by the nonlocal elasticity and verified through molecular simulation results. The value of the scale coefficient in the nonlocal theory is recommended to be about 0.7 nm for the application of the nonlocal theory in analysis of CNTs. Arash and Wang[27] present a review of the application of nonlocal elastic models in modeling of CNTs and graphenes. Explicit expressions for the critical buckling load and strain were derived for axially loaded rods/tubes based on the nonlocal Timoshenko beam theory.[21,30–32] Wang *et al.*[21,30,31] developed nonlocal elastic beam and shell models to investigate the small-scale effect on buckling analysis of CNTs under compression.

The nonlocal Timoshenko beam model was implemented[33–35] to investigate an axial buckling of SWCNTs embedded in an elastic medium, and the influence of the small-scale effect and stiffness of the surrounding medium was investigated. Zhang *et al.*[36,37] presented an assessment of nonlocal beam and shell models in the prediction of critical buckling strains of axially loaded SWCNTs by use of molecular dynamic (MD) simulation results for SWCNTs.

3. Basic Equations of Nonlocal Timoshenko Beam and the Method of Initial Values

Basic equations of classical Timoshenko beam are given by[21]

$$\frac{dv}{dz} = \frac{1}{\kappa GF} T - \Omega, \tag{1}$$

$$\frac{d\Omega}{dz} = \frac{M}{EI}, \tag{2}$$

$$\frac{dM}{dz} = T, \tag{3}$$

$$\frac{dT}{dz} = P\,\frac{d^2 v}{dz^2}, \tag{4}$$

where z is the longitudinal coordinate, v is the transverse displacement, Ω is the rotation due to bending, M is bending moment, T is the shear force, P is an axial compressive load, G is the shear modulus, E is the elasticity modulus, F is the cross-sectional area of the beam and I is the second moment of area. The shear correction factor (κ) is given for circular and rectangular cross-sections as follows:[38]

$$\kappa = \frac{6(1+\nu)}{7+6\nu}, \quad \kappa = \frac{10(1+\nu)}{12+11\nu}. \tag{5}$$

The following relation exists between local and nonlocal stresses.

$$\sigma_{\text{nonloc}}(z) - \gamma^2 \sigma''_{\text{nonloc}}(z) = \sigma_{\text{loc}}(z), \tag{6}$$

where γ is nonlocality parameter. If the two sides of Eq. (6) are multiplied by $y\,dF$ and integrated over the cross-section, the following relation is obtained:

$$\int_F \sigma_{\text{nonloc}}(z) y\,dF - \gamma^2 \int_F \sigma''_{\text{nonloc}}(z) y\,dF = \int_F \sigma_{\text{loc}}(z) y\,dF. \tag{7}$$

By modifying this equation, one obtains

$$M_{\text{nonloc}}(z) - \gamma^2 M''{}_{\text{nonloc}}(z) = M_{\text{loc}}(z). \tag{8}$$

By using Eq. (8), Eqs. (1)–(4) take the following form:

$$\frac{dv}{dz} = \frac{1}{FG\kappa} T - \Omega, \tag{9}$$

$$\frac{d\Omega}{dz} = \frac{P - FG\kappa}{FG\kappa(\gamma^2 P - EI) + EI\,P} M, \tag{10}$$

$$\frac{dM}{dz} = T, \tag{11}$$

$$\frac{dT}{dz} = \frac{FG\kappa P}{FG\kappa(\gamma^2 P - EI) + EI\,P} M. \tag{12}$$

The system of equations given above is the basic equations of nonlocal Timoshenko beam subjected to an axial compressive load.

(a) Equations (9)–(12) revert to the basic equations of nonlocal Euler beam in the limit κ goes to infinity.
(b) Equations (9)–(12) revert to the basic equations of classical Timoshenko beam in the limit γ goes to zero.
(c) Equations (9)–(12) revert to the basic equations of classical Euler beam in the limit γ goes to zero and κ goes to infinity.

The system of equations given by (9)–(12) can be written in matrix form as follows:

$$\frac{d}{dz}\begin{pmatrix} v \\ \Omega \\ M \\ T \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & \dfrac{1}{FG\kappa} \\ 0 & 0 & \dfrac{P - FG\kappa}{EIP + FG(P\gamma^2 - EI)\kappa} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{FGP\kappa}{EIP + FG(P\gamma^2 - EI)\kappa} & 0 \end{pmatrix} \begin{pmatrix} v \\ \Omega \\ M \\ T \end{pmatrix}, \tag{13}$$

where

$$\vec{y} = \begin{pmatrix} v \\ \Omega \\ M \\ T \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 & 0 & \dfrac{1}{FG\kappa} \\ 0 & 0 & \dfrac{P - FG\kappa}{EIP + FG(P\gamma^2 - EI)\kappa} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{FGP\kappa}{EIP + FG(P\gamma^2 - EI)\kappa} & 0 \end{pmatrix}. \tag{14}$$

The initial value problem can be written in the following form:

$$\frac{d\vec{y}}{dz} = A\vec{y}, \quad \vec{y}_0 = \begin{pmatrix} v(0) \\ \Omega(0) \\ M(0) \\ T(0) \end{pmatrix} = \begin{pmatrix} v_0 \\ \Omega_0 \\ M_0 \\ T_0 \end{pmatrix}. \tag{15}$$

Solution of initial value problem given by Eq. (15) is

$$\vec{y}(z) = TM(z)\vec{y}_0 \tag{16}$$

(see Ref. 39), where $TM(z)$ is called as carry-over matrix or transverse matrix. $TM(z)$ is given below:

$$TM(z,\alpha,EI) = \begin{pmatrix} 1 & -z & \dfrac{\cos(z\alpha) - 1}{P} & \dfrac{EI\alpha\sin(z\alpha) - Pz(\alpha^2\gamma^2 + 1)}{EIP\alpha^2} \\ 0 & 1 & \dfrac{(\alpha^2\gamma^2 + 1)\sin(z\alpha)}{EI\alpha} & -\dfrac{(\alpha^2\gamma^2 + 1)(\cos(z\alpha) - 1)}{EI\alpha^2} \\ 0 & 0 & \cos(z\alpha) & \dfrac{\sin(z\alpha)}{\alpha} \\ 0 & 0 & -\alpha\sin(z\alpha) & \cos(z\alpha) \end{pmatrix}, \tag{17}$$

where

$$\alpha = \frac{\sqrt{F}\sqrt{G}\sqrt{\kappa}\sqrt{P}}{\sqrt{FG\kappa(EI - \gamma^2 P) - EI\,P}}. \tag{18}$$

4. Examples

4.1. *Calculation of critical buckling load of rods/tubes with pinned ends*

In this case, two of the initial values are known (see Fig. 1).

$$v_0 = 0, \quad M_0 = 0. \tag{19}$$

End conditions are

$$v(L) = 0, \quad M(L) = 0. \tag{20}$$

The remaining two initial values will be found with the help of end conditions. The solution of the problem can be written as follows with the aid of Eq. (16).

$$v(z) = TM_{12}(z)\Omega_0 + TM_{14}(z)T_0 \tag{21}$$

$$\Omega(z) = TM_{22}(z)\Omega_0 + TM_{24}(z)T_0 \tag{22}$$

$$M(z) = TM_{32}(z)\Omega_0 + TM_{34}(z)T_0 \tag{23}$$

$$T(z) = TM_{42}(z)\Omega_0 + TM_{44}(z)T_0 \tag{24}$$

With the help of the end conditions, the buckling determinant is obtained as

$$\begin{vmatrix} TM_{12}(L) & TM_{14}(L) \\ TM_{32}(L) & TM_{34}(L) \end{vmatrix} = -\frac{L\sin(\alpha L)}{\alpha} = 0 \Rightarrow \alpha L = \pi \tag{25}$$

Critical buckling load of nonlocal Timoshenko beam (P) is obtained as follows (see Eq. (18)):

$$P = \frac{\pi^2 EI}{L^2\left(\frac{\pi^2 EI}{FG\kappa L^2} + \frac{\pi^2\gamma^2}{L^2} + 1\right)}. \tag{26}$$

(a) The critical load given by Eq. (26) returns to the critical load of the classical Timoshenko beam (P_1) in the limit γ goes to zero.

$$P_1 = \frac{EI\pi^2}{L^2 + \frac{EI\pi^2}{FG\kappa}}. \tag{27}$$

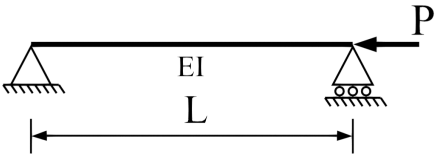

Fig. 1. Beam with pinned ends.

(b) The critical load given by Eq. (26) returns to the critical load of the classical Euler beam (P_2) in the limit γ goes to zero and κ goes to infinity.

$$P_2 = \frac{\pi^2 EI}{L^2}. \tag{28}$$

(c) The critical load given by Eq. (26) returns to the critical load of the nonlocal Euler beam (P_3) in the limit κ goes to infinity.

$$P_3 = \frac{\pi^2 EI}{\pi^2\gamma^2 + L^2}. \tag{29}$$

(see Ref. 21).

4.2. *Calculation of critical buckling load of rods/tubes with clamped-free ends*

In this case, two of the initial values are known (see Fig. 2).

$$v_0 = 0, \quad \Omega_0 = 0. \tag{30}$$

End conditions are[21]

$$T(z) - P\frac{dv}{dz}\bigg|_{z=L} = 0, \quad M(L) = 0. \tag{31}$$

The remaining two initial values will be found with the help of end conditions. The solution of the problem can be written as follows with the aid of Eq. (16).

$$v(z) = TM_{13}(z)M_0 + TM_{14}(z)T_0, \tag{32}$$

$$\Omega(z) = TM_{23}(z)M_0 + TM_{24}(z)T_0, \tag{33}$$

$$M(z) = TM_{33}(z)M_0 + TM_{34}(z)T_0, \tag{34}$$

$$T(z) = TM_{43}(z)M_0 + TM_{44}(z)T_0. \tag{35}$$

With the help of the end conditions, the buckling determinant is obtained as

$$\begin{vmatrix} TM_{43}(L) - P\,TM'_{13}(L) & TM_{44}(L) - P\,TM'_{14}(L) \\ TM_{33}(L) & TM_{34}(L) \end{vmatrix} = -\frac{P(\alpha^2\gamma^2+1)\cos(\alpha L)}{\alpha^2 EI} = 0 \Rightarrow \alpha L = \frac{\pi}{2}. \tag{36}$$

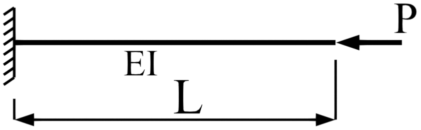

Fig. 2. Beam with clamped-free ends.

Critical buckling load of nonlocal Timoshenko beam (P) is obtained as follows (see Eq. (18))

$$P = \frac{\pi^2 EI}{L^2\left(\frac{\pi^2 EI}{FG\kappa L^2} + \frac{\pi^2\gamma^2}{L^2} + 4\right)}. \tag{37}$$

(a) The critical load given by Eq. (37) returns to the critical load of the classical Timoshenko beam (P_1) in the limit γ goes to zero.

$$P_1 = \frac{\pi^2 EI}{\frac{\pi^2 EI}{FG\kappa} + 4L^2}. \tag{38}$$

(b) The critical load given by Eq. (27) returns to the critical load of the classical Euler beam (P_2) in the limit γ goes to zero and κ goes to infinity.

$$P_2 = \frac{\pi^2 EI}{4L^2}. \tag{39}$$

(c) The critical load given by Eq. (27) returns to the critical load of the nonlocal Euler beam (P_3) in the limit κ goes to infinity.

$$P_3 = \frac{\pi^2 EI}{\pi^2\gamma^2 + 4L^2}. \tag{40}$$

4.3. *Calculation of critical buckling load of rods/tubes with variable cross-section*

In the beam shown in Fig. 3, two of the initial values are known.

$$v_0 = 0, \quad M_0 = 0 \tag{41}$$

End conditions are

$$v(L_1 + L_2) = 0, \quad M(L_1 + L_2) = 0. \tag{42}$$

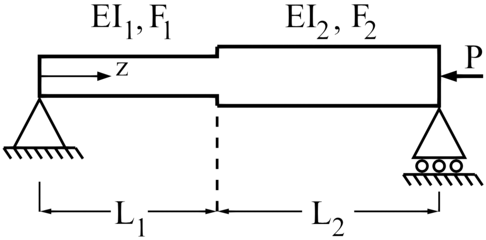

Fig. 3. Variable cross-section beam.

The vector $\vec{y}$ at $z = L_1$ and $z = L_2$ can be written as follows:

$$\begin{aligned}\vec{y}(L_1) &= TM(L_1, \alpha_1, EI_1)\vec{y_0}\\ \vec{y}(L_1 + L_2) &= TM(L_2, \alpha_2, EI_2)\vec{y}(L_1)\\ &= TM(L_2, \alpha_2, EI_2)\,TM(L_1, \alpha_1, EI_1)\vec{y}_0.\end{aligned} \tag{43}$$

The matrix *TTM* is defined as

$$TTM = TM(L_2, \alpha_2, EI_2)\,TM(L_1, \alpha_1, EI_1), \tag{44}$$

where

$$TM(z, \alpha_i, EI_i) = \begin{pmatrix} 1 & -z & \dfrac{\cos(z\alpha_i) - 1}{P} & \dfrac{EI_i\alpha_i \sin(z\alpha_i) - Pz(\gamma^2\alpha_i^2 + 1)}{PEI_i\alpha_i^2} \\ 0 & 1 & \dfrac{(\gamma^2\alpha_i^2 + 1)\sin(z\alpha_i)}{EI_i\alpha_i} & -\dfrac{(\gamma^2\alpha_i^2 + 1)(\cos(z\alpha_i) - 1)}{EI_i\alpha_i^2} \\ 0 & 0 & \cos(z\alpha_i) & \dfrac{\sin(z\alpha_i)}{\alpha_i} \\ 0 & 0 & \alpha_i(-\sin(z\alpha_i)) & \cos(z\alpha_i) \end{pmatrix}. \tag{45}$$

The buckling determinant is obtained as

$$\begin{vmatrix} TTM_{12} & TTM_{14} \\ TTM_{32} & TTM_{34} \end{vmatrix} = -\frac{\begin{gathered}(L_1 + L_2)(\alpha_1 \sin(\alpha_2 L_2)\cos(\alpha_1 L_1) \\ +\ \alpha_2 \sin(\alpha_1 L_1)\cos(\alpha_2 L_2))\end{gathered}}{\alpha_1\alpha_2} = 0, \tag{46}$$

where the parameter α is as follows:

(a) For nonlocal Timoshenko beam is

$$\alpha_i = \frac{\sqrt{G}\sqrt{\kappa}\sqrt{P}\sqrt{F_i}}{\sqrt{G\kappa F_i(EI_i - \gamma^2 P) - P\,EI_i}}. \tag{47}$$

(b) For nonlocal Euler beam is

$$\alpha_i = \frac{\sqrt{G}\sqrt{P}\sqrt{F_i}}{\sqrt{-GF_i(\gamma^2 P - EI_i)}}. \tag{48}$$

(c) For local Timoshenko beam is

$$\alpha_i = \frac{\sqrt{G}\sqrt{\kappa}\sqrt{P}\sqrt{F_i}}{\sqrt{-EI_i(P - G\kappa F_i)}}. \tag{49}$$

(d) For local Euler beam is

$$\alpha_i = \frac{\sqrt{P}}{\sqrt{EI_i}}. \tag{50}$$

The dimensionless critical loads are given in Figs. 4 and 5 for various values of the parameters. ($\alpha = \frac{b_1}{b_2}$, $\omega_i = \frac{h_i}{L_i}$, $\beta = \frac{h_1}{h_2}$, $\theta = \frac{L_1}{L_2}$, $\eta_i = \frac{b_i}{h_i}$, $\xi = \frac{\gamma}{L}$, b_i is the width of the cross-section and h_i is the height of the cross-section.)

(P_{DLE}: dimensionless critical load of local Euler beam, P_{DLT}: dimensionless critical load of local Timoshenko beam, P_{DNE}: dimensionless critical load of nonlocal Euler beam, P_{DNT}: dimensionless critical load of nonlocal Timoshenko beam).

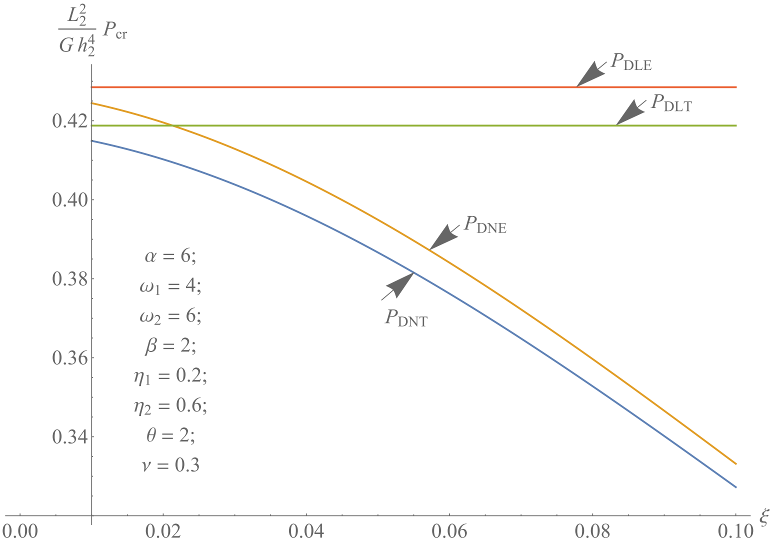

Fig. 4. Dimensionless critical loads of variable cross-section beam.

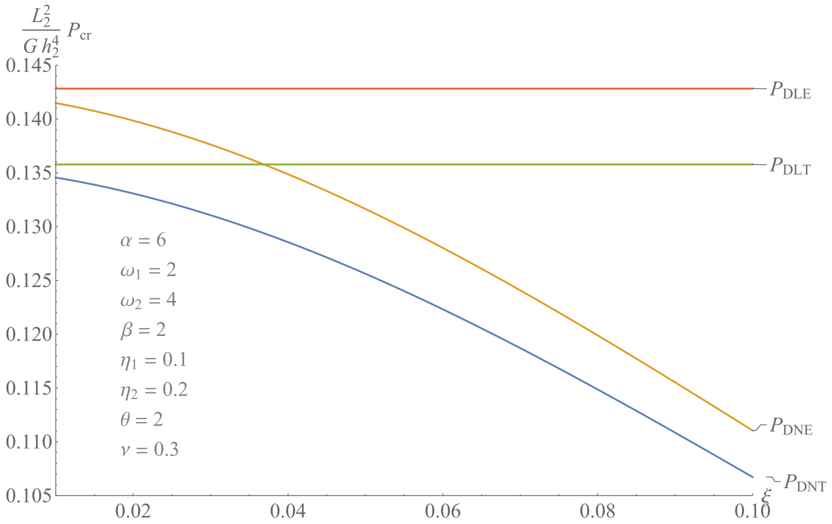

Fig. 5. Dimensionless critical loads of variable cross-section beam.

4.3.1. *Special case*

The buckling determinant given in Eq. (46) reverts to the determinant in Eq. (25) for $L_1 = L_2 = L/2$, $F_1 = F_2$ and $EI_1 = EI_2$.

4.4. *Calculation of critical buckling load of Gerber beam*

Gerber beam is made by joining simple beams with hinges and removing the remaining freedoms by supports (see Fig. 6).

In the Gerber beam shown in Fig. 6, the following values are known:

$$v(0) = 0,\; M(0) = 0, \quad M(L_1) = 0, \quad v(L) = 0,\, \Omega(L) = 0. \tag{51}$$

That is

$$\vec{y}(0) = \begin{pmatrix} 0 \\ \Omega(0) \\ 0 \\ T(0) \end{pmatrix}, \quad \vec{y}({L_1}^-) = \begin{pmatrix} v(L_1) \\ \Omega({L_1}^-) \\ 0 \\ T(L_1) \end{pmatrix}, \quad \vec{y}({L_1}^+) = \begin{pmatrix} v(L_1) \\ \Omega({L_1}^-) + \lambda \\ 0 \\ T(L_1) \end{pmatrix}. \tag{52}$$

Using Eq. (16), the following relations can be written:

$$\vec{y}({L_1}^-) = TM(L_1).\vec{y}(0), \tag{53a}$$

$$\vec{y}(L) = TM(L_2).\vec{y}({L_1}^+) = TM(L_2).(\vec{y}({L_1}^-) + \vec{u}), \tag{53b}$$

where

$$\vec{u} = \begin{pmatrix} 0 \\ \lambda \\ 0 \\ 0 \end{pmatrix}. \tag{54}$$

Using Eqs. (53a) and (53b), the following relations can be written:

$$\begin{aligned} \vec{y}(L) &= TM(L_2).\vec{y}({L_1}^-) + TM(L_2).\vec{u} \\ &= TM(L_2).TM(L_1).\vec{y}(0) + TM(L_2).\vec{u} \\ &= TM(L).\vec{y}(0) + TM(L_2).\vec{u}, \end{aligned} \tag{55}$$

$$M(L_1) = 0, \quad v(L) = 0, \quad \Omega(L) = 0. \tag{56}$$

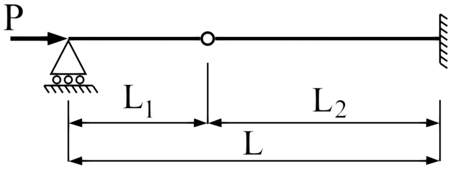

Fig. 6. Gerber beam.

The following relations are obtained by the conditions given in Eq. (56).

$$\begin{aligned} v(L) &= -L_2\lambda + \frac{T_0(\alpha EI\sin(\alpha L) - LP(\alpha^2\gamma^2+1))}{\alpha^2 EIP} - L\,\Omega(0) = 0, \\ \Omega(L) &= -\frac{T_0(\alpha^2\gamma^2+1)(\cos(\alpha L)-1)}{\alpha^2 EI} + \lambda + \Omega(0) = 0, \\ M(L_1) &= \frac{T_0\sin(L_1\alpha)}{\alpha} = 0. \end{aligned} \tag{57}$$

The buckling determinant can be written as follows:

$$\mathrm{Det}\begin{pmatrix} -L_2 & -L & \dfrac{EI\alpha\sin(L\alpha) - LP(\alpha^2\gamma^2+1)}{EIP\alpha^2} \\ 1 & 1 & -\dfrac{(\alpha^2\gamma^2+1)(\cos(L\alpha)-1)}{EI\alpha^2} \\ 0 & 0 & \dfrac{\sin(L_1\alpha)}{\alpha} \end{pmatrix} = \frac{(L-L_2)\sin(\alpha L_1)}{\alpha} = 0. \tag{58}$$

Critical buckling load of nonlocal Timoshenko beam is obtained as

$$P_{\mathrm{NT}} = \frac{\pi^2 EI}{{L_1}^2\left(\frac{\pi^2 EI}{FG\kappa {L_1}^2} + \frac{\pi^2\gamma^2}{{L_1}^2} + 1\right)}. \tag{59}$$

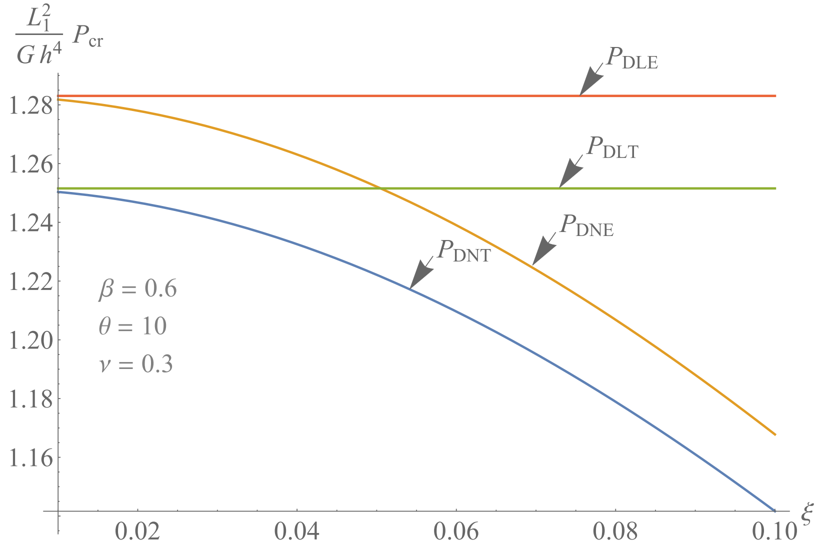

Fig. 7. Dimensionless critical loads of Gerber beam.

Critical buckling load of nonlocal Euler beam is

$$P_{\mathrm{NE}} = \lim_{\kappa \to \infty} P_{\mathrm{NT}} = \frac{\pi^2 EI}{\pi^2 \gamma^2 + {L_1}^2}. \tag{60}$$

Critical buckling load of local Timoshenko beam is

$$P_{\mathrm{LT}} = \frac{\pi^2 EI}{{L_1}^2 \left(\frac{\pi^2 EI}{FG\kappa {L_1}^2} + 1 \right)}. \tag{61}$$

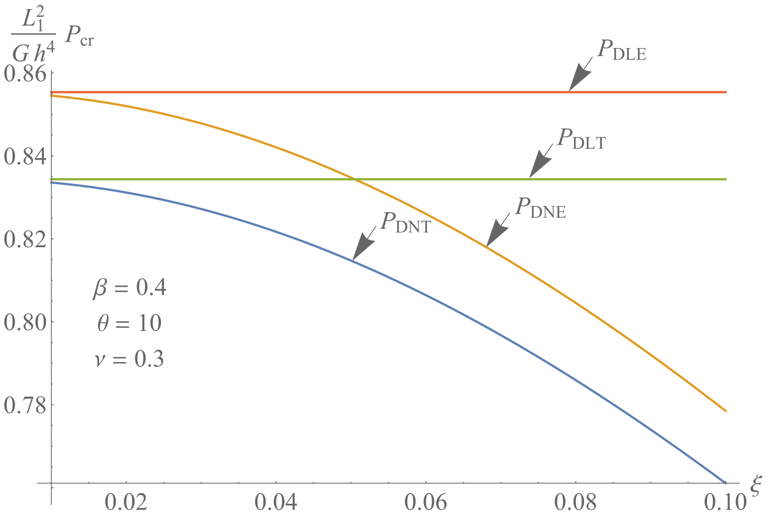

Fig. 8. Dimensionless critical loads of Gerber beam.

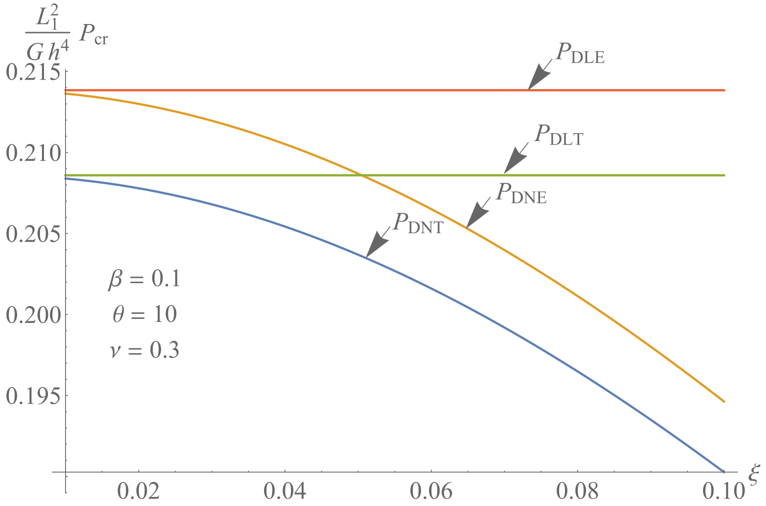

Fig. 9. Dimensionless critical loads of Gerber beam.

Critical buckling load of local Euler beam is

$$P_{\mathrm{LE}} = \lim_{\gamma \to 0} P_{\mathrm{NE}} = \frac{\pi^2 EI}{{L_1}^2}. \tag{62}$$

The dimensionless critical loads are given in Figs. 7–12 for various values of the parameters. (P_{DLE}: critical buckling load of dimensionless local Euler beam, P_{DLT}: critical buckling load of dimensionless local Timoshenko beam, P_{DNE}: critical

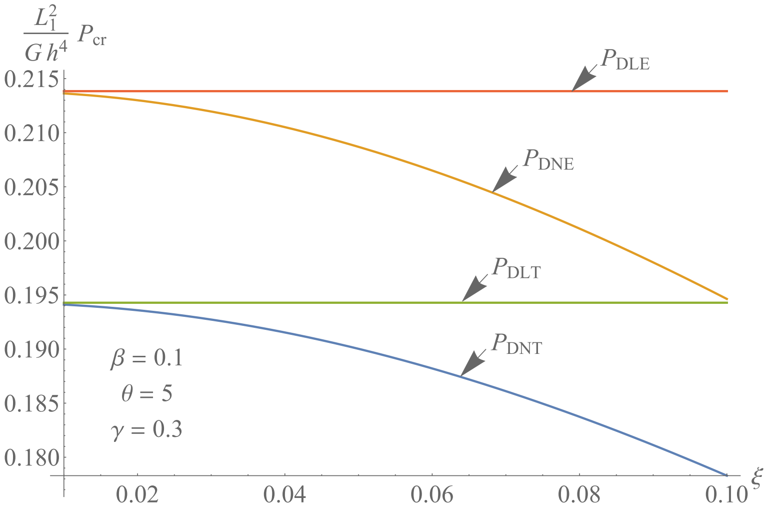

Fig. 10. Dimensionless critical loads of Gerber beam.

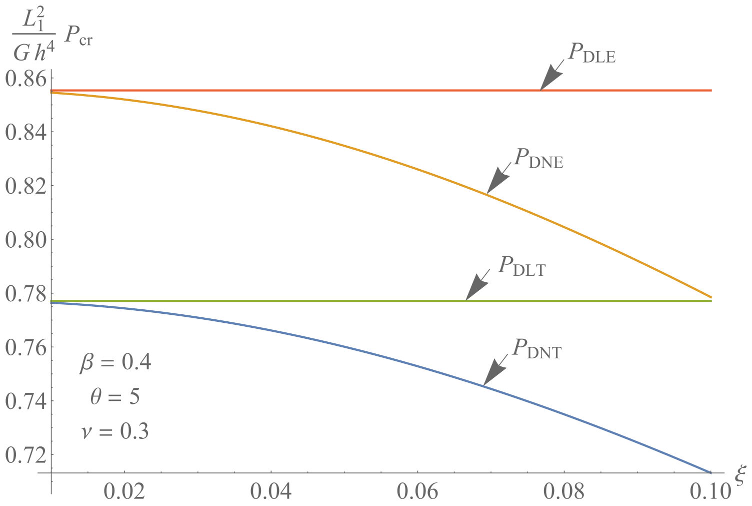

Fig. 11. Dimensionless critical loads of Gerber beam.

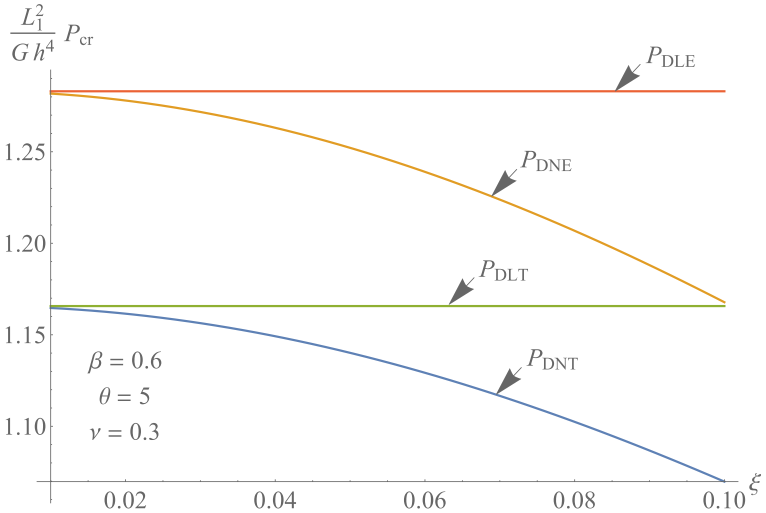

Fig. 12. Dimensionless critical loads of Gerber beam.

buckling load of dimensionless nonlocal Euler beam, P_{DNT}: critical buckling load of dimensionless nonlocal Timoshenko beam.)

Dimensionless parameters are selected as follows:

$$\xi = \frac{\gamma}{L_1}, \quad \beta = \frac{b}{h}, \quad \theta = \frac{h}{L_1}, \tag{63}$$

where b and h are the dimensions of the beam cross-section.

5. Concluding Remarks

In this work, based on Eringen's nonlocal elasticity theory, the buckling of Timoshenko beam with the method of initial values is studied. The nonlocal Timoshenko theory considers both the scale effect and the effect of transverse shear deformation. For this reason, it has a significant role in the investigation of nanoscale elements. A priority of the method is that the high degree of statical indeterminacy adds no further hardship to the solution of the problem.[22] It is interesting to note that the size of the matrix that gives the buckling determinant is 2 × 2. If the conventional technique applies to these problems the buckling loads are found by the determinant of 4 × 4 matrix (see Sec. 4.1 and 4.2). For the problems given in Secs. 4.3 and 4.4, the buckling determinant was obtained with the help of 3 × 3 determinants. 8 × 8 determinants arise in the solution of the same problems with the conventional technique. Nonlocal theories do not neglect small-scale effects, for this reason, the critical loads calculated by local and non-local theories are different for large values of γ/L (see Figs. 4–12). In the nonlocal Timoshenko beam, both the scale effect and the shear effect are considered. For this reason, the safest critical loads are those belonging to the nonlocal Timoshenko beam (see Figs. 4–12).

Acknowledgement

This research is supported by the Alexander von Humboldt Foundation.

References

1. B. Yang, G. Xiong, K. Ding, S. Nie, W. Zhang, Y. Hu and G. Dai, Experimental and numerical studies on lateral-torsional buckling of gj structural steel beams under a concentrated loading condition, *Int. J. Struct. Stab. Dynam.* **16**(1) (2016) 1640004.
2. H. Wu, S. Kitipornchai and J. Yang, Free vibration and buckling analysis of sandwich beams with functionally graded carbon nanotube-reinforced composite face sheets, *Int. J. Struct. Stab. Dynam.* **15**(7) (2015) 1540011.
3. R. Ansari and R. Gholami, Size-dependent buckling and postbuckling analyses of first-order shear deformable magneto-electro-thermo elastic nanoplates based on the nonlocal elasticity theory, *Int. J. Struct. Stab. Dynam.* **17**(1) (2017) 1750014.
4. K. L. Ekinci and M. L. Roukes, Nanoelectromechanical systems, *Rev. Sci. Instrum.* **76**(6) (2005) 061101.
5. R. Gholami, R. Ansari, A. Darvizeh and S. Sahmani, Axial buckling and dynamic stability of functionally graded microshells based on the modified couple stress theory, *Int. J. Struct. Stab. Dynam.* **15**(4) (2015) 1450070.
6. N. V. Lavrik, M. J. Sepaniak and P. G. Datskos, Cantilever transducers as a platform for chemical and biological sensors, *Rev. Sci. Instrum.* **75**(7) (2004) 2229–2253.
7. A. C. Eringen, Nonlocal polar elastic continua, *Int. J. Eng. Sci.* **10**(1) (1972) 1–16.
8. A. C. Eringen and D. G. B. Edelen, On nonlocal elasticity, *Int. J. Eng. Sci.* **10**(3) (1972) 233–248.
9. J. Peddieson, An application of the micropolar fluid model to the calculation of a turbulent shear flow, *Int. J. Eng. Sci.* **10**(1) (1972) 23–32.
10. A. C. Eringen, On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves, *J. Appl. Phys.* **54**(9) (1983) 4703–4710.
11. Y. S. Al Rjoub and A. G. Hamad, Free vibration of functionally euler-bernoulli and timoshenko graded porous beams using the transfer matrix method, *KSCE J. Civil Eng.* **21**(3) (2017) 792–806.
12. R. Artan and A. Tepe, The initial values method for buckling of nonlocal bars with application in nanotechnology, *Eur. J. Mech. —A/Solids* **27**(3) (2008) 469–477.
13. R. Barretta, L. Feo, R. Luciano and F. Marotti de Sciarra, Variational formulations for functionally graded nonlocal bernoulli-euler nanobeams, *Composite Struct.* **129** (2015) 80–89.
14. M. A. Eltaher, M. E. Khater and Samir A. Emam, A review on nonlocal elastic models for bending, buckling, vibrations, and wave propagation of nanoscale beams, *Appl. Math. Modell.* **40**(5–6) (2016) 4109–4128.
15. S. A. H. Hosseini and O. Rahmani, Surface effects on buckling of double nanobeam system based on nonlocal timoshenko model, *Int. J. Struct. Stab. Dynam.* **16**(10) (2016) 1550077.
16. C. W. Lim and R. Xu, Analytical solutions for coupled tension-bending of nanobeam-columns considering nonlocal size effects, *Acta Mechanica* **223**(4) (2012) 789–809.
17. C. Liu, L. L. Ke, Y. S. Wang, J. Yang and S. Kitipornchai, Buckling and post-buckling of size-dependent piezoelectric timoshenko nanobeams subject to thermo-electro-mechanical loadings, *Int. J. Struct. Stab. Dynam.* **14**(3) (2014) 1350067.
18. Q. Wang, Wave propagation in carbon nanotubes via nonlocal continuum mechanics, *J. Appl. Phys.* **98**(12) (2005) 124301.

19. Q. Wang and V. K. Varadan, Vibration of carbon nanotubes studied using nonlocal continuum mechanics, *Smart Mater. Struct.* **15**(2) (2006) 659.
20. L. Wang and H. Hu, Flexural wave propagation in single-walled carbon nanotubes, *Phys. Rev. B* **71**(19) (2005) 195412.
21. C. M. Wang, Y. Y. Zhang, S. S. Ramesh and S. Kitipornchai, Buckling analysis of micro- and nano-rods/tubes based on nonlocal timoshenko beam theory, *J. Phys. D: Appl. Phys.* **39**(17) (2006) 3904.
22. R. Artan, The analytical calculation of circular rods of variable cross-section by the initial values method, *Comput. Struct.* **62**(3) (1997) 445–461.
23. I. A. Kunin, *Elastic Media with Microstructure: One-dimensional Models*, Elastic Media with Microstructure (Springer, 1982).
24. K. Tserpes and N. Silvestre, *Modeling of Carbon Nanotubes, Graphene and their Composites* (Springer Series in Materials Science. Springer, Cham, 2014).
25. W. H. Duan, C. M. Wang and Y. Y. Zhang, Calibration of nonlocal scaling effect parameter for free vibration of carbon nanotubes by molecular dynamics, *J. Appl. Phys.* **101**(2) (2007) 024305.
26. Q. Wang, Q. K. Han and B. C. Wen, Estimate of material property of carbon nanotubes via nonlocal elasticity, *Adv. Theor. Appl. Mech.* **1**(1) (2008) 1–10.
27. B. Arash and Q. Wang, *A Review on the Application of Nonlocal Elastic Models in Modeling of Carbon Nanotubes and Graphenes* (Springer, International Publishing, Cham, 2014), pp. 57–82.
28. F. Khademolhosseini, A. S. Phani, A. Nojeh and N. Rajapakse, Nonlocal continuum modeling and molecular dynamics simulation of torsional vibration of carbon nanotubes, *IEEE Trans. Nanotechnol.* **11**(1) (2012) 34–43.
29. Y. J. Liang and Q. Han, Prediction of nonlocal scale parameter for carbon nanotubes, *Sci. China Phys. Mech. Astronomy* **55**(9) (2012) 1670–1678.
30. Q. Wang, V. K. Varadan and S. T. Quek, Small scale effect on elastic buckling of carbon nanotubes with nonlocal continuum models, *Phys. Lett. A* **357** (2006) 130–135.
31. Q. Wang, G. Y. Zhou and K. C. Lin, Scale effect on wave propagation of double-walled carbon nanotubes, *Int. J. Solids Struct.* **43**(20) (2006) 6071–6084.
32. Y. Q. Zhang, G. R. Liu and J. S. Wang, Small-scale effects on buckling of multiwalled carbon nanotubes under axial compression, *Phys. Rev. B* **70** (2004) 205430.
33. T. Murmu and S. C. Pradhan, Buckling analysis of a single-walled carbon nanotube embedded in an elastic medium based on nonlocal elasticity and timoshenko beam theory and using dqm, *Physica E: Low-dimensional Syst. Nanostruct.* **41**(7) (2009) 1232–1239.
34. T. Murmu and S. C. Pradhan, Buckling of biaxially compressed orthotropic plates at small scales, *Mech. Res. Commun.* **36**(8) (2009) 933–938.
35. T. Murmu and S. C. Pradhan, Thermo-mechanical vibration of a single-walled carbon nanotube embedded in an elastic medium based on nonlocal elasticity theory, *Comput. Mater. Sci.* **46**(4) (2009) 854–859.
36. Y. Y. Zhang, C. M. Wang, W. H. Duan, Y. Xiang and Z. Zong, Assessment of continuum mechanics models in predicting buckling strains of single-walled carbon nanotubes, *Nanotechnology* **20**(39) (2009) 395707.
37. Y.-Y. Zhang, C. Wang and V. B. C. Tan, Assessment of timoshenko beam models for vibrational behavior of single-walled carbon nanotubes using molecular dynamics, *Adv. Appl. Math. Mech.* **1**(2) (2009) 89–106.
38. G. R. Cowper, The shear coefficient in timoshenko's beam theory, *J.ASME Appl. Mech.* **33**(2) (1966) 335–340.
39. R. Artan and A. Toksöz, Stability analysis of gradient elastic beams by the method of initial value, *Arch. Appl. Mech.* **83**(8) (2013) 1129–1144.

View publication stats